

# On varieties of Hilbert type

Lior Bary-Soroker, Arno Fehm and Sebastian Petersen

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## Abstract

A variety  $X$  over a field  $K$  is of Hilbert type if  $X(K)$  is not thin. We prove that if  $f: X \rightarrow S$  is a dominant morphism of  $K$ -varieties and both  $S$  and all fibers  $f^{-1}(s)$ ,  $s \in S(K)$ , are of Hilbert type, then so is  $X$ . We apply this to answer a question of Serre on products of varieties and to generalize a result of Colliot-Thélène and Sansuc on algebraic groups.

## 1 Introduction

In the terminology of thin sets (we recall this notion in Section 2), Hilbert's irreducibility theorem asserts that  $\mathbb{A}_K^n(K)$  is not thin, for any number field  $K$  and any  $n \geq 1$ . As a natural generalization a  $K$ -variety  $X$  is called of Hilbert type if  $X(K)$  is not thin. The importance of this definition stems from the observation of Colliot-Thélène and Sansuc [2] that the inverse Galois problem would be settled if every unirational variety over  $\mathbb{Q}$  was of Hilbert type.

In this direction, Colliot-Thélène and Sansuc [2, Corollary 7.15] prove that any connected reductive algebraic group over a number field is of Hilbert type. This immediately raises the question whether the same holds for all linear algebraic groups (note that these are unirational). Another question, asked by Serre [15, p. 21], is whether a product of two varieties of Hilbert type is again of Hilbert

type. The main result of this paper gives a sufficient condition for a variety to be of Hilbert type:

**Theorem 1.1.** *Let  $K$  be a field and  $f: X \rightarrow S$  a dominant morphism of  $K$ -varieties. Assume that the set of  $s \in S(K)$  for which the fiber  $f^{-1}(s)$  is a  $K$ -variety of Hilbert type is not thin. Then  $X$  is of Hilbert type.*

As an immediate consequence we get the following result for a family of varieties over a variety of Hilbert type:

**Corollary 1.2.** *Let  $K$  be a field and  $f: X \rightarrow S$  a dominant morphism of  $K$ -varieties. Assume that  $S$  is of Hilbert type and that for every  $s \in S(K)$  the fiber  $f^{-1}(s)$  is of Hilbert type. Then  $X$  is of Hilbert type.*

Using this result we resolve both questions discussed above affirmatively, see Corollary 3.4 and Proposition 4.2.

## 2 Background

Let  $K$  be a field. A  $K$ -variety is a separated scheme of finite type over  $K$  which is geometrically reduced and geometrically irreducible. Thus, an open subscheme of a  $K$ -variety is again a  $K$ -variety. If  $f: X \rightarrow S$  is a morphism of  $K$ -varieties and  $s \in S(K)$ , then  $f^{-1}(s) := X \times_S \text{Spec}(\kappa(s))$ , where  $\kappa(s)$  is the residue field of  $s$ , denotes the scheme theoretic fiber of  $f$  at  $s$ . This fiber is a separated scheme of finite type over  $K$ , which needs not be reduced or connected in general. We identify the set  $f^{-1}(s)(K)$  of  $K$ -rational points of the fiber with the set theoretic fiber  $\{x \in X(K) \mid f(x) = s\}$ .

Let  $X$  be a  $K$ -variety. A subset  $T$  of  $X(K)$  is called *thin* if there exists a proper Zariski-closed subset  $C$  of  $X$ , a finite set  $I$ , and for each  $i \in I$  a  $K$ -variety  $Y_i$  with  $\dim Y_i = \dim X$  and a dominant separable morphism  $p_i: Y_i \rightarrow X$  of degree  $\geq 2$  such that

$$T \subseteq \bigcup_{i \in I} p_i(Y_i(K)) \cup C(K).$$

A  $K$ -variety  $X$  is of *Hilbert type* if  $X(K)$  is not thin, cf. [15, Definition 3.1.2]. Note that  $X$  is of Hilbert type if and only if some (or every) open subscheme of  $X$  is of Hilbert type, cf. [15, p. 20]. A field  $K$  is *Hilbertian* if  $\mathbb{A}_K^1$  is of Hilbert type. We note that if there exists a  $K$ -variety  $X$  of positive dimension such that  $X$  is of Hilbert type, then  $K$  is Hilbertian [5, Proposition 13.5.3].

### 3 Proof of Theorem 1.1

A key tool in the proof of Theorem 1.1 is the following consequence of Stein factorization.

**Lemma 3.1.** *Let  $K$  be a field and  $\psi: Y \rightarrow S$  a dominant morphism of normal  $K$ -varieties. Then there exists a nonempty open subscheme  $U \subset S$ , a  $K$ -variety  $T$  and a factorization*

$$\psi^{-1}(U) \xrightarrow{g} T \xrightarrow{r} U$$

*of  $\psi$  such that the fibers of  $g$  are geometrically irreducible and  $r$  is finite and étale.*

*Proof.* See [10, Lemma 9]. □

**Lemma 3.2.** *Let  $K$  be a field and  $f: X \rightarrow S$  a dominant morphism of normal  $K$ -varieties. Assume that the set  $\Sigma$  of  $s \in S(K)$  for which  $f^{-1}(s)$  is a  $K$ -variety of Hilbert type is not thin. Let  $I$  be a finite set and let  $p_i: Y_i \rightarrow X$ ,  $i \in I$ , be finite étale morphisms of degree  $\geq 2$ . Then  $X(K) \not\subset \bigcup_{i \in I} p_i(Y_i(K))$ .*

*Proof.* For  $i \in I$  consider the composite morphism  $\psi_i := f \circ p_i: Y_i \rightarrow S$ . By Lemma 3.1 there is a nonempty open subscheme  $U_i$  of  $S$  and a factorization

$$\psi_i^{-1}(U_i) \xrightarrow{g_i} T_i \xrightarrow{r_i} U_i$$

of  $\psi_i$  such that the morphism  $g_i$  has geometrically irreducible fibers,  $r_i$  is finite and étale, and such that  $T_i$  is a  $K$ -variety. We now replace successively  $S$  by  $\bigcap_{i \in I} U_i$ ,  $X$  by  $f^{-1}(S)$ ,  $T_i$  by  $r_i^{-1}(S)$  and  $Y_i$  by  $p_i^{-1}(X)$ , to assume in addition that  $r_i: T_i \rightarrow S$  is finite étale for every  $i \in I$ .

For  $s \in S(K)$  denote by  $X_s := f^{-1}(s)$  the fiber of  $f$  over  $s$ . Then  $X_s$  is a  $K$ -variety of Hilbert type for each  $s \in \Sigma$ . Furthermore we define  $Y_{i,s} := \psi_i^{-1}(s)$  and let  $p_{i,s}: Y_{i,s} \rightarrow X_s$  be the corresponding projection morphism. Then  $p_{i,s}$  is a finite étale morphism of the same degree as  $p_i$ . In particular, the  $K$ -scheme  $Y_{i,s}$  is geometrically reduced. For every  $s \in S(K)$  and every  $i \in I$  we have constructed a commutative diagram

$$\begin{array}{ccccc} Y_{i,s} & \longrightarrow & Y_i & \xrightarrow{g_i} & T_i \\ p_{i,s} \downarrow & & p_i \downarrow & \searrow \psi_i & \downarrow r_i \\ X_s & \longrightarrow & X & \xrightarrow{f} & S \end{array}$$

in which the left hand rectangle is cartesian. Set  $J := \{i \in I : \deg(r_i) \geq 2\}$ . Then  $\bigcup_{i \in J} r_i(T_i(K)) \subseteq S(K)$  is thin, so by assumption there exists

$$s \in \Sigma \setminus \bigcup_{i \in J} r_i(T_i(K)).$$

For  $i \in J$  there is no  $K$ -rational point of  $T_i$  over  $s$ , hence  $Y_{i,s}(K) = \emptyset$  for every  $i \in J$ . For  $i \in I \setminus J$ , the finite étale morphism  $r_i$  is of degree 1, hence an isomorphism, and therefore  $Y_{i,s}$  is geometrically irreducible. Thus,  $Y_{i,s}$  is a  $K$ -variety. So since  $X_s$  is of Hilbert type, there exists  $x \in X_s(K)$  such that  $x \notin \bigcup_{i \in I \setminus J} p_{i,s}(Y_{i,s}(K))$ . Thus

$$x \notin \bigcup_{i \in J \setminus I} p_{i,s}(Y_{i,s}(K)) = \bigcup_{i \in I} p_{i,s}(Y_{i,s}(K)),$$

hence  $x \notin \bigcup_{i \in I} p_i(Y_i(K))$ , as needed.  $\square$

We shall use the following well-known fact.

**Lemma 3.3.** *Let  $K$  be a field, let  $X, Y$  be  $K$ -varieties with  $\dim X = \dim Y$ , and let  $p: Y \rightarrow X$  be a dominant separable morphism. Then there exists a nonempty open subscheme  $U$  of  $X$  such that the restriction of  $p$  to a morphism  $p^{-1}(U) \rightarrow U$  is finite and étale.*

*Proof of Theorem 1.1.* Let  $K$  be a field, and  $f: X \rightarrow S$  a dominant morphism of  $K$ -varieties. Assume that the set  $\Sigma$  of those  $s \in S(K)$  for which  $f^{-1}(s)$  is of Hilbert type is not thin. Let  $C \subseteq X$  be a proper Zariski-closed subset. Let  $I$  be a finite set and suppose that  $Y_i$  is a  $K$ -variety with  $\dim(Y_i) = \dim(X)$  and

$p_i: Y_i \rightarrow X$  is a dominant separable morphism of degree  $\geq 2$ , for every  $i \in I$ . We have to show that  $X(K) \not\subseteq C(K) \cup \bigcup_{i \in I} p_i(Y_i(K))$ .

By Lemma 3.3 and [8, 6.12.6, 6.13.5] there exists a normal nonempty open subscheme  $X' \subset X \setminus C$  such that the restriction of each  $p_i$  to a morphism  $p_i^{-1}(X') \rightarrow X'$  is finite and étale. The image  $f(X')$  contains a nonempty open subscheme  $S'$  of  $S$  (cf. [7, 1.8.4], [6, 9.2.2]). Furthermore,  $S'$  contains a nonempty normal open subscheme  $S''$ . Let us define  $X'' := f^{-1}(S'') \cap X'$  and  $Y_i'' := p_i^{-1}(X'')$ . Then the restriction of  $f$  to a morphism  $f'': X'' \rightarrow S''$  is a surjective morphism of normal  $K$ -varieties,  $\Sigma \cap S''(K)$  is not thin, and  $f''^{-1}(s)$  is of Hilbert type for every  $s \in \Sigma \cap S''(K)$  because it is an open subscheme of  $f^{-1}(s)$ . The restriction  $p_i''$  of  $p_i$  to a morphism  $Y_i'' \rightarrow X''$  is finite and étale for every  $i \in I$ . By Lemma 3.2 applied to  $f''$  and the  $p_i''$  we have

$$\begin{aligned} \emptyset &\neq X''(K) \setminus \bigcup_{i \in I} p_i''(Y_i''(K)) \\ &= X''(K) \setminus \bigcup_{i \in I} p_i(Y_i(K)) \\ &\subseteq X(K) \setminus \left( C(K) \cup \bigcup_{i \in I} p_i(Y_i(K)) \right), \end{aligned}$$

so  $X(K) \not\subseteq C(K) \cup \bigcup_{i \in I} p_i(Y_i(K))$ , as needed.  $\square$

As an immediate consequence we get an affirmative solution of Serre's question mentioned in the introduction.

**Corollary 3.4.** *Let  $K$  be a field. If  $X, Y$  are  $K$ -varieties of Hilbert type, then  $X \times Y$  is of Hilbert type.*

*Proof.* Denote by  $f: X \times Y \rightarrow X$  the projection. Then  $f^{-1}(x)$  is isomorphic to  $Y$  and hence of Hilbert type for every  $x \in X(K)$ . Thus Corollary 1.2 gives that  $X \times Y$  is of Hilbert type.  $\square$

## 4 Algebraic groups of Hilbert type

By an *algebraic group* over a field  $K$  we shall mean a connected smooth group scheme over  $K$ . Recall that such an algebraic group is a  $K$ -variety, see [9, Exp VI<sub>A</sub>, 0.3, 2.1.2, 2.4]. If  $G$  is an algebraic group over  $K$ , then  $G(K_s)$  is a  $\text{Gal}(K)$ -group, where  $K_s$  denotes a separable closure of  $K$  and  $\text{Gal}(K) = \text{Gal}(K_s/K)$  is the absolute Galois group of  $K$ , and there is an associated Galois cohomology pointed set  $H^1(K, G) = H^1(\text{Gal}(K), G(K_s))$ , which classifies isomorphism classes of  $G(K_s)$ -torsors, cf. [12, Prop. 1.2.4].

**Proposition 4.1.** *Let  $K$  be a field and let*

$$1 \rightarrow N \rightarrow G \xrightarrow{p} Q \rightarrow 1$$

*be a short exact sequence of algebraic groups over  $K$ . If  $H^1(K, N) = 1$  and both  $N$  and  $Q$  are of Hilbert type, then  $G$  is of Hilbert type.*

*Proof.* It suffices to show that  $p^{-1}(x)$  is of Hilbert type for every  $x \in Q(K)$ , because then Corollary 1.2 implies the assertion. Let  $x \in Q(K)$  and  $F = p^{-1}(x)$ . There is an exact sequence of  $\text{Gal}(K)$ -groups

$$1 \rightarrow N(K_s) \rightarrow G(K_s) \rightarrow Q(K_s) \rightarrow 1,$$

where the right hand map is surjective, because for every point  $x \in Q(K_s)$  the fiber over  $x$  is a non-empty  $K_s$ -variety and thus has a  $K_s$ -rational point. Since the  $\text{Gal}(K)$ -set  $F(K_s)$  is a coset of  $N(K_s)$ , it is a  $N(K_s)$ -torsor. Our hypothesis  $H^1(K, N) = 1$  implies that  $F(K_s)$  is isomorphic to the trivial  $N(K_s)$ -torsor  $N(K_s)$ . It follows that  $F$  is isomorphic to  $N$  as a  $K$ -variety, hence  $F$  is of Hilbert type.  $\square$

Using this, we generalize the result of Colliot-Thélène and Sansuc [2, Corollary 7.15] from reductive groups to arbitrary linear groups.

**Theorem 4.2.** *Every linear algebraic group  $G$  over a perfect Hilbertian field  $K$  is of Hilbert type.*

*Proof.* We denote by  $G_u$  the unipotent radical of  $G$  (cf. [11, Proposition XVII.1.2]). We have a short exact sequence of algebraic groups over  $K$

$$1 \rightarrow G_u \rightarrow G \rightarrow Q \rightarrow 1 \quad (*)$$

with  $Q$  reductive, cf. [11, Proposition XVII.2.2]. By [2, Corollary 7.15],  $Q$  is of Hilbert type. Since  $K$  is perfect,  $G_u$  is split, i.e. there exists a series of normal algebraic subgroups

$$1 = U_0 \subseteq \cdots \subseteq U_n = G_u$$

such that  $U_{i+1}/U_i \cong \mathbb{G}_a$  for each  $i$ , cf. [1, 15.5(ii)]. The groups  $U_i$  are unipotent, hence  $H^1(K, U_i) = 1$  by [14, Ch. III.2.1, Prop. 6], and  $\mathbb{G}_a$  is of Hilbert type since  $K$  is Hilbertian. Thus, an inductive application of Proposition 4.1 implies that  $G_u$  is of Hilbert type. Finally we apply Proposition 4.1 to the exact sequence  $(*)$  to conclude that  $G$  is of Hilbert type.  $\square$

*Remark 4.3.* The special case of Theorem 4.2 where  $G$  is simply connected and  $K$  is finitely generated is also a consequence of a result of Corvaja, see [3, Corollary 1.7].

*Remark 4.4.* We point out that Theorem 4.2 could be deduced also from Corollary 3.4 (instead of 1.1) via [13, Corollary 1] and the fact that a unipotent group over a perfect field is rational, cf. [9, XIV, 6.3].

As a consequence of Theorem 4.2, we get a more general statement for homogeneous spaces, which was pointed out to us by Borovoi:

**Corollary 4.5.** *If  $G$  is a linear algebraic group over a perfect Hilbertian field  $K$ , and  $H$  is algebraic subgroup of  $G$ , then the quotient  $G/H$  is of Hilbert type.*

*Proof.* For the existence of the quotient  $Q := G/H$  see for example [1, Chapter II Theorem 6.8]. If  $\mathcal{H}$  denotes the generic fiber of  $G \rightarrow Q$  and  $\bar{F}$  is an algebraic closure of the function field  $K(Q)$  of  $Q$ , then  $\mathcal{H}_{\bar{F}} \cong H_{\bar{F}}$  by translation on  $G$ . Thus,  $\mathcal{H}$  is geometrically irreducible since  $H$  is, so [2, Proposition 7.13] implies that  $Q$  is of Hilbert type.  $\square$

We also get a complete classification of the algebraic groups that are of Hilbert type over a number field:

**Corollary 4.6.** *An algebraic group  $G$  over a number field  $K$  is of Hilbert type if and only if it is linear.*

*Proof.* If  $G$  is linear, then it is of Hilbert type by Theorem 4.2. Conversely, assume that  $G$  is of Hilbert type. Chevalley's theorem [4, Theorem 1.1] gives a short exact sequence of algebraic groups over  $K$ ,

$$1 \rightarrow H \rightarrow G \rightarrow A \rightarrow 1$$

with  $H$  linear and  $A$  an abelian variety. As in the proof of Corollary 4.5 we conclude that the generic fiber of  $G \rightarrow A$  is geometrically irreducible, and therefore  $A$  is of Hilbert type. Since no nontrivial abelian variety over a number field is of Hilbert type, cf. [5, Remark 13.5.4],  $A$  is trivial and  $G \cong H$  is linear.  $\square$

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